

A CUSP SLOPE-CENTRAL ANISOTROPY THEOREM

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ABSTRACT

For a wide class of self-gravitating systems, we show that if the density is cusped like $r^{-\gamma}$ near the center, then the limiting value of the anisotropy parameter $\beta = 1 - \langle v_r^2 \rangle / (2\langle v_t^2 \rangle)$ at the center may not be greater than $\gamma/2$. Here $\langle v_r^2 \rangle$ and $\langle v_t^2 \rangle$ are the radial and tangential velocity second moments. This follows from the nonnegativity of the phase-space density. We compare this theorem to other proposed relations between the cusp slope and the central anisotropy to clarify their applicabilities and underlying assumptions. The extension of this theorem to tracer populations in an externally imposed potential is also derived. In particular, for stars moving in the vicinity of a central black hole, this reduces to $\gamma \geq \beta + \frac{1}{2}$, indicating that an isotropic system in Keplerian potential should be cusped at least as steep as $r^{-1/2}$. Similar limits have been noticed before for specific forms of the distribution function, but here we establish this as a general result.

Subject headings: stellar dynamics — methods: analytical — galaxies: kinematics and dynamics

1. INTRODUCTION

Numerical simulations of halo formation provide strong evidence that the inner parts of dark matter halos are strongly cusped. Typically, the density profile ρ behaves like $r^{-\gamma}$, where γ lies between 1 and 1.5 (Navarro, Frenk, & White 1995; Moore et al. 1998). Although this numerical result seems well established, observational evidence that dark halos are cusped has been surprisingly elusive. A disparate body of data – including the rotation curves of dwarf spiral galaxies (Palunas & Williams 2000; de Blok et al. 2001), the kinematics of Local Group dwarf spheroidal galaxies (Kleyna et al. 2003), and mass models of gravitational lens systems (Tyson, Kochanski, & dell’Antonio 1998) – seem to favor constant-density cores.

Recently, Hansen (2004) claimed that the only density slopes permitted by the spherically symmetric and isotropic Jeans equations are $1 \leq \gamma \leq 3$ if the phase-space density-like quantity, $\rho/\langle v^2 \rangle^{3/2}$, follows a power law (Taylor & Navarro 2001). This result was inferred from the condition that the power-law solution of the Jeans and Poisson equations is physical, subject to the “equation of state” (EOS), $\rho \propto r^{-p} \langle v_r^2 \rangle^c$, where p and c are constants. He further argued that, for the system with an anisotropic velocity dispersion tensor, $1 + \beta \leq \gamma \leq 3$. Here the anisotropy parameter is $\beta = 1 - \langle v_r^2 \rangle / (2\langle v_t^2 \rangle)$, where $\langle v_r^2 \rangle$ and $\langle v_t^2 \rangle$ are the radial and the tangential velocity second moments (Binney & Tremaine 1987). Given the nearly isotropic conditions found in the central parts of simulated dark halos, this already seems to indicate that the density profile cannot be shallower than $\rho \sim r^{-1}$.

The idea of looking for constraints between the central density slope and the anisotropy is an excellent one. In § 2, we show that the inequality $\gamma \geq 2\beta$ is a necessary condition for the nonnegativity of the distribution function (DF). This generalizes two well-established results: (1) a spherical system with a hole in the center cannot be supported by an isotropic velocity dispersion tensor (i.e., $\gamma \geq 2\beta = 0$; see Tremaine et al. 1994), and (2) a spherical system with particles in purely radial orbits cannot be supported by a density

cusp shallower than the isothermal cusp (i.e., $\gamma \geq 2\beta = 2$; see Richstone & Tremaine 1984). In § 3, we study the scale-free power-law cusps that inspired Hansen (2004) and show that the inequality $\gamma \geq 1 + \beta$ is related to the boundary condition at infinity rather than at the center. Finally, in § 4, we consider the generalization of the theorem to tracer populations in an externally imposed cusped potential – for example, to stars moving around a central black hole. In particular, with the potential cusped as $r^{-\delta}$ ($0 \leq \delta \leq 1$), we find that the inequality becomes $\gamma \geq 2\beta + (\frac{1}{2} - \beta)\delta$. While a similar limit has been noted in the literature for the case in which the DF is scale-free (White 1981; de Bruijne, van der Marel, & de Zeeuw 1996), we derive the limit as a general condition for a DF with a centrally diverging (and not necessarily self-consistent) potential.

2. PROOF OF A CUSP SLOPE-ANISOTROPY THEOREM

Here we shall prove that if the density is cusped like $r^{-\gamma}$ near the center, then the limiting value of β at the center may not be greater than $\gamma/2$. This relation is indeed suggested by solving the Jeans equation for constant β . That is, the one-dimensional radial velocity dispersion obtained as the solution, in general, diverges at the center if $\gamma < 2\beta$, which is unphysical unless the central potential well depth is infinite. In fact, we will show that the inequality $\gamma \geq 2\beta$ is the necessary condition for the nonnegativity of the DF.

2.1. Constant Anisotropy Distribution Functions

First, let us suppose that the DF is given by the *Ansatz*³

$$f(E, L) = L^{-2\beta} f_E(E), \quad (1)$$

where L is the specific angular momentum, and $f_E(E)$ is a function of the binding energy E alone. This seems as though it is a restrictive assumption, but this is not really the case. Rather, equation (1) arises naturally as the most divergent term in a Laurent series expansion with respect to L at $L = 0$ for a very wide class of DFs. By integrating equation (1) over

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³ Strictly speaking, the *Ansatz* is valid only for $\beta < 1$. However, it can be extended to $\beta = 1$ using the relation $\lim_{\beta \rightarrow 1} [L^{2\beta} / \Gamma(1 - \beta)] = \delta(L^2)$, where $\delta(x)$ is the Dirac delta “function.” Subsequently, equations (2)–(5) are still valid for $f(E, L) = \delta(L^2) f_E(E)$ as the $\beta = 1$ limit simply without the $\Gamma(1 - \beta)$ factor.

velocity space, we find that the density is given by

$$\rho = r^{-2\beta} D_\beta \int_0^\psi (\psi - E)^{1/2-\beta} f_E(E) dE, \quad (2)$$

$$D_\beta = \frac{(2\pi)^{3/2} \Gamma(1-\beta)}{2^\beta \Gamma(3/2-\beta)}, \quad (3)$$

where ψ is the relative potential and $\Gamma(x)$ is the gamma function. The unknown function $f_E(E)$ can be found from the formula (Cuddeford 1991; Kochanek 1996; Wilkinson & Evans 1999; An & Evans 2006; Evans & An 2006)

$$f_E(E) = C_\beta \left[\int_0^E \frac{d\psi}{(E-\psi)^\alpha} \frac{d^{n+1}h}{d\psi^{n+1}} + \frac{1}{E^\alpha} \frac{d^n h}{d\psi^n} \right]_{\psi=0} \quad (4)$$

$$C_\beta = \frac{2^\beta}{(2\pi)^{3/2} \Gamma(1-\alpha) \Gamma(1-\beta)}. \quad (5)$$

Here $h = r^{2\beta} \rho$ is expressed as a function of ψ , and $n = \lfloor \frac{3}{2} - \beta \rfloor$ and $\alpha = \frac{3}{2} - \beta - n$ are the integer floor and the fractional part of $\frac{3}{2} - \beta$. It is a simple exercise to show that the anisotropy parameter for this model is the same as the parameter β in the expression of DF (eq. 1).

By considering the limit of equation (2) as $r \rightarrow 0$ one can infer that ρ should diverge at least as fast as $r^{-2\beta}$ if $\beta > 0$ and cannot approach zero faster than $r^{2(-\beta)}$ if $\beta < 0$ unless the integral vanishes in the same limit. This in fact is the basic argument that leads to the theorem. In the following, we provide a more stringent proof of the theorem.

2.1.1. The Case $\beta \leq \frac{1}{2}$

For $\beta < \frac{1}{2}$, a direct generalization of the proof of Tremaine et al. (1994) for isotropic models suffices. That is, from equation (2), we find that

$$\frac{d}{d\psi} (r^{2\beta} \rho) = \tilde{D}_\beta \int_0^\psi \frac{f_E(E) dE}{(\psi - E)^{1/2+\beta}} \geq 0, \quad (6)$$

$$\tilde{D}_\beta = \left(\frac{1}{2} - \beta \right) D_\beta = \frac{(2\pi)^{3/2} \Gamma(1-\beta)}{2^\beta \Gamma(1/2-\beta)} \quad (7)$$

for any physical DF, as the integrand is always positive. Similarly, if $\beta = \frac{1}{2}$, then

$$\frac{d}{d\psi} (r\rho) = 2\pi^2 \frac{d}{d\psi} \int_0^\psi f_E(E) dE = 2\pi^2 f_E(\psi) \geq 0. \quad (8)$$

However,

$$\frac{dh}{d\psi} = \frac{dr}{d\psi} \frac{h}{r} \frac{d \ln h}{d \ln r} = \frac{r^{2\beta-1} \rho}{d\psi/dr} \left(2\beta + \frac{d \ln \rho}{d \ln r} \right). \quad (9)$$

Since $d\psi/dr \leq 0$ for any physical potential, we thus find that

$$2\beta \leq -\frac{d \ln \rho}{d \ln r}. \quad (10)$$

This holds everywhere. Specializing to the limit at the center, we obtain the desired result.

2.1.2. The Case $\beta > \frac{1}{2}$

When $\beta > \frac{1}{2}$, equation (6) is invalid, and therefore a different proof is required. For this purpose, we first note that

$$\frac{1}{(\psi - E)^{\beta-1/2}} > \frac{1}{\psi^{\beta-1/2}} > 0 \quad (11)$$

for $0 < E < \psi$ and $\beta > \frac{1}{2}$. Then from equation (2), we find that

$$\rho > \frac{D_\beta}{r^{2\beta} \psi^{\beta-1/2}} \int_0^\psi f_E(E) dE. \quad (12)$$

Therefore,

$$\lim_{r \rightarrow 0} (\rho r^{2\beta} \psi^{\beta-1/2}) \geq D_\beta \int_0^{\psi_0} f_E(E) dE > 0, \quad (13)$$

where $\psi_0 = \psi(r=0)$. Next, if we consider the case in which ψ_0 is finite (see § 4 for the centrally diverging potential), we have

$$\lim_{r \rightarrow 0} (\rho r^{2\beta} \psi^{\beta-1/2}) = \psi_0^{\beta-1/2} \lim_{r \rightarrow 0} (\rho r^{2\beta}) > 0. \quad (14)$$

Since $\psi_0 > 0$, we find that $\lim_{r \rightarrow 0} (\rho r^{2\beta}) > 0$, that is, $\lim_{r \rightarrow 0} h$ either is nonzero positive and finite or diverges to positive infinity. If the former is the case, it is straightforward to show that (using l'Hôpital's rule)

$$\lim_{r \rightarrow 0} h = \lim_{r \rightarrow 0} \frac{hr}{r} = \lim_{r \rightarrow 0} \frac{d(hr)}{dr} = \lim_{r \rightarrow 0} h \frac{d \ln(hr)}{d \ln r}, \quad (15)$$

and consequently that

$$\lim_{r \rightarrow 0} \frac{d \ln(hr)}{d \ln r} = 1 \quad \Rightarrow \quad \lim_{r \rightarrow 0} \frac{d \ln h}{d \ln r} = 0. \quad (16)$$

On the other hand, if $\lim_{r \rightarrow 0} h$ is divergent, l'Hôpital's rule indicates that

$$\lim_{r \rightarrow 0} \frac{d \ln h}{d \ln r} = \lim_{r \rightarrow 0} \frac{\ln h}{\ln r}. \quad (17)$$

However, we have $(\ln h / \ln r) < 0$ for sufficiently small r , so

$$\lim_{r \rightarrow 0} \frac{\ln h}{\ln r} \leq 0 \quad \Rightarrow \quad \lim_{r \rightarrow 0} \frac{d \ln h}{d \ln r} \leq 0. \quad (18)$$

Hence, for both cases,

$$\lim_{r \rightarrow 0} \frac{d \ln h}{d \ln r} = \lim_{r \rightarrow 0} \frac{d \ln \rho}{d \ln r} + 2\beta \leq 0 \quad (19)$$

$$\Rightarrow \gamma = -\lim_{r \rightarrow 0} \frac{d \ln \rho}{d \ln r} \geq 2\beta, \quad (20)$$

which is the desired result.

2.2. General (Analytic) Distribution Functions

This result is in fact far more general than the assumed *Ansatz* for the DF (eq. 1). For example, Ciotti & Pellegrini (1992) found the same limit ($\gamma \geq 2\beta = 0$) for the Osipkov (1979a,b)-Merritt (1985) type DF, which is manifestly not in the form of equation (1). We extend the limit derived in the preceding section to a wide class of DFs by the following simple argument. In general, any analytic DF can be expressed either as sums of equation (1) or in terms of a Laurent series expansion with respect to L at $L = 0$ (really a special class of the former). Then since $L = rv_T$, as $r \rightarrow 0$, the DF is dominated by the term associated with the leading order of L , and consequently so is the behavior of the density near the center. It is also straightforward to show that the anisotropy parameter at the center is indeed determined by β_0 , where ‘ $-2\beta_0$ ’ is the power to the leading term of L . The desired result therefore follows from the preceding proof for the special case of the DF with a single term. In Appendix, we discuss the conditions of applicability of the proof more mathematically and argue that the theorem holds for all physically reasonable DFs of spherical systems.

We find that the cusp slope–anisotropy theorem ($\gamma \geq 2\beta$) is actually quite reasonable. It implies that, if the anisotropy is

radially biased ($\beta > 0$) near the center, the density is cusped, and that, unless the cusp slope is steeper than that of isothermal cusp ($\gamma = 2$), there is a finite upper limit to β that is strictly smaller than unity. Similarly, if the density is cored ($\gamma = 0$), the central anisotropy is either tangentially biased or at most isotropic.

3. SCALE-FREE DENSITY PROFILE

Recently, Hansen (2004) derived a similar but stricter inequality $\gamma \geq 1 + \beta$, based on the condition that there exists a physical power-law solution to the spherically symmetric Jeans and Poisson equations with the EOS, $\rho \propto r^{-p} \langle v_r^2 \rangle^c$. Since $\beta \leq 1$, his result is stronger than our result. However, we note that his result is strictly valid only if both ρ and $\langle v_r^2 \rangle$ behave as the pure power law extending globally to the infinity. In fact, we find that the condition $\gamma \geq \beta + 1$ originates from the boundary condition at infinity rather than at the center and thus argue that the result should be understood to refer to the asymptotic density power index at infinity, not the central density slope. In particular, the supposed piece of evidence cited by Hansen (2004) for his inequality (Łokas & Hoffman 2000; see also Bouvier & Janin 1968) is in fact due to the constraint imposed by $\gamma \geq 2\beta$ on the central slope through the positivity of the DF (§ 2; see also Richstone & Tremaine 1984), since it involves the case of purely radial motion ($\beta = 1$) for which the two limits coincide ($\gamma \geq 2$).

The general integral solution of the Jeans equation with constant β can be written by admitting an integration constant \tilde{B} ,

$$r^{2\beta} \rho \langle v_r^2 \rangle = \tilde{B} + \int_{r_0}^r dr' r'^{2\beta} \rho(r') \frac{d\psi}{dr} \Big|_{r=r'}. \quad (21)$$

The potential gradient (i.e., the local gravitational acceleration) can be found from the enclosed mass

$$\frac{d\psi}{dr} = -\frac{GM_r}{r^2}; \quad M_r = M_{r_0} + 4\pi \int_{r_0}^r \rho(r') r'^2 dr', \quad (22)$$

where the negative sign is due to our choice of sign for ψ . If we assume a strict power-law behavior for the density as $\rho = Ar^{-\gamma}$, where $A > 0$ is constant, then M_r is given by ($\gamma \neq 3$)

$$M_r = M_{r_0} + 4\pi A(3 - \gamma)^{-1} (r^{3-\gamma} - r_0^{3-\gamma}). \quad (23)$$

If the power law extends to the center, the result must be valid for the choice of $r_0 = 0$. However, then the mass within any finite radius M_r always diverges for $\gamma \geq 3$ even if M_0 is finite, and therefore the model is unphysical. The resulting upper limit $\gamma < 3$ is well established. By substituting equation (22) and $\rho = Ar^{-\gamma}$ in equation (21), $\langle v_r^2 \rangle$ as a function of r ($\gamma \neq 3$, $\gamma \neq \beta + 1$, and $\gamma \neq 2\beta - 1$) is found to be

$$\begin{aligned} \langle v_r^2 \rangle = & Br^{\gamma-2\beta} - r^{2-\gamma} \frac{2\pi GA}{(3-\gamma)(\beta-\gamma+1)} \\ & + \frac{1}{r} \left[\frac{4\pi GA r_0^{3-\gamma}}{(3-\gamma)(2\beta-\gamma-1)} - \frac{GM_{r_0}}{2\beta-\gamma-1} \right], \end{aligned} \quad (24)$$

where B is an integration constant to be determined from the boundary condition. Here if we assume strictly scale-free behavior for ρ , equation (24) is valid everywhere extending from $r = 0$ to $r = \infty$. With $r_0 = 0$ and $M_0 = 0$, we can show that the condition for $\langle v_r^2 \rangle$ to be nonnegative everywhere leads to the inequality $\gamma > \beta + 1$, as found by Hansen (2004), and $B \geq 0$. In addition, the self-similarity implies strict power-law behavior for $\langle v_r^2 \rangle$ as well. Since $A > 0$, this can only be obtained with the choice of $B = 0$. In fact, the choice of $B = 0$ can independently be deduced from the boundary condition at infinity.

That is, $\langle v_r^2 \rangle$ is nondivergent for a finite potential ($\gamma < 2$), or if the potential diverges ($\gamma > 2$), the velocity dispersion cannot diverge faster than the potential does.

However, if we relax the assumption that ρ is strictly scale-free everywhere and replace it with ρ being locally well approximated by the power law near the center, equation (24) is valid *only* for the region where $\rho \approx Ar^{-\gamma}$, so the condition that $\langle v_r^2 \rangle$ is nonnegative only needs to be checked for this regime. Provided that the power law provides a good approximation to the behavior of ρ near the center and that we limit our attention to a self-consistent density-potential system, it is reasonable to choose $r_0 = 0$ and $M_0 = 0$. Then for $\gamma > 2\beta$, with any positive constant B , equation (24) returns the correct behavior of the velocity dispersion near the center, although its validity does not extend to infinity. For $\gamma > \beta + 1$, the velocity dispersion ($r_0 = M_0 = 0$) is given by

$$\langle v_r^2 \rangle = r^{2-\gamma} \left[\frac{2\pi GA}{(3-\gamma)(\gamma-\beta-1)} + Br^{2(\gamma-\beta-1)} \right]. \quad (25)$$

Near the center, we find that $\langle v_r^2 \rangle \sim A'r^{2-\gamma}$, where A' is a positive constant. This is just the approximate local power-law behavior of the velocity dispersion near the center (valid locally for any B). In addition, if $\beta + 1 < \gamma \leq 2$, the central velocity dispersion is finite. Although $\langle v_r^2 \rangle$ diverges at the center if $\gamma > 2 \geq \beta + 1$, this behavior can be physical because the self-consistent potential well depth for this case is also infinite. On the other hand, if $\gamma < \beta + 1$, the same velocity dispersion can be written as

$$\langle v_r^2 \rangle = r^{\gamma-2\beta} \left[B - r^{2(\beta+1-\gamma)} \frac{2\pi GA}{(3-\gamma)(\beta+1-\gamma)} \right]. \quad (26)$$

In other words, since $\gamma - 2\beta < 2 - \gamma$ if $\gamma < \beta + 1$, the leading term for the velocity dispersion near the center is given by $\langle v_r^2 \rangle \sim Br^{\gamma-2\beta}$ provided that $B \neq 0$. While not strictly scale-free, the local behavior of the velocity dispersion near the center can still be approximated as a power law, and furthermore, as long as $B > 0$ and $0 \leq \gamma - 2\beta < 2 - \gamma$, it is positive and finite. While we note that for a sufficiently large r , equation (26) eventually becomes negative, since the $r^{2-\gamma}$ term becomes dominant as $r \rightarrow \infty$, this does not restrict the central power index for density, provided that the behavior of ρ starts to deviate from $\sim Ar^{-\gamma}$ (toward the steeper falloff) at smaller r than the value at which $\langle v_r^2 \rangle = 0$ in equation (26).

Let us consider the explicit example of the Hernquist (1990) model, which has a r^{-1} density cusp (i.e., $\gamma = 1$). The radial velocity dispersion of the constant- β model is (see, e.g., Baes & Dejonghe 2002; Evans & An 2005)

$$\langle v_r^2 \rangle = \frac{GM}{(5-2\beta)} \frac{r}{(r+a)^2} {}_2F_1\left(1, 5; 6-2\beta; \frac{a}{r+a}\right), \quad (27)$$

where the potential is given by $\psi = GM/(r+a)$. It is straightforward to show that, for $0 < \beta \leq \frac{1}{2}$, $\langle v_r^2 \rangle$ is everywhere positive finite with its leading order behavior near the center given by $\sim r^{1-2\beta}$ (if $\beta = \frac{1}{2}$, then $\langle v_r^2 \rangle = \psi/4$; see also Evans & An 2005). On the other hand, if $\beta < 0$, the leading order for $\langle v_r^2 \rangle$ near the center is found to be $\sim r$ with a positive coefficient. For the isotropic case ($\beta = 0$), the elementary functional expression for the velocity dispersion is given in equation (10) of Hernquist (1990), whose leading order behavior is found to be $\sim r \ln r^{-1}$ (see, e.g., eq. 11 of Hernquist 1990).

As another example, we consider the dark matter profile proposed by Dehnen & McLaughlin (2005). They solved the spherically symmetric Jeans and Poisson equations with the same EOS as in Hansen (2004). However, they found a family

of “realistic” models with a finite mass and an infinite extent. With $\rho \propto r^{-p} \langle v_r^2 \rangle^{3/2}$, the inner density cusp of their models is given by $\gamma = (7 + 10\beta_0)/9$, where β_0 is the anisotropy parameter at the center. It is clear that for all members of these models $\gamma < \beta_0 + 1$, since $\beta_0 \leq 1 < 2$, thus violating the inequality $\gamma \geq \beta + 1$. On the other hand, our result $\gamma \geq 2\beta$ indicates that they are physical if and only if $\beta_0 \leq \frac{7}{8}$. In fact, $\langle v_r^2 \rangle$ near the center for this family behaves as $\sim r^{(7-8\beta_0)/9}$, and thus the limit $\gamma \geq 2\beta$ is equivalent to the condition that the central velocity dispersion is finite.

A similar analysis of equation (24) can be applied to discover the asymptotic behavior of the velocity dispersion at infinity. Suppose that ρ asymptotically approaches a power law for a sufficiently large r . Then for the same range, equation (24) is a valid expression for $\langle v_r^2 \rangle$, provided that the power law $\sim A\rho^{-\gamma}$ describes the asymptotic behavior of ρ and M_{r_0} is the mass within r_0 . Here $B = 0$ from the boundary condition at infinity. If the total mass is finite ($\gamma > 3 > \beta + 1 > 2\beta - 1$), we can simply set $r_0 = \infty$ and $M_\infty = M$, where M is the total mass. Then since $-1 > 2 - \gamma$, the asymptotic behavior of the velocity dispersion is given by $\langle v_r^2 \rangle \sim (\gamma + 1 - 2\beta)^{-1} GM/r$, which is consistent with Keplerian falloff. For an infinite-mass system, we find that $2 - \gamma > -1$, and thus the leading term of $\langle v_r^2 \rangle$ for $r \rightarrow \infty$ is $\sim A'r^{2-\gamma}$. Here A' is a positive constant if $\gamma > \beta + 1$, whereas it is a negative constant if $\gamma < \beta + 1$. In other words, from the condition of nonnegativity of the velocity dispersion toward infinity, we actually recover the inequality of Hansen (2004), $\gamma \geq \beta + 1$, although here γ and β are the asymptotic density power index and the anisotropy parameter at *infinity*. Although the velocity dispersion diverges as $r \rightarrow \infty$ if $\gamma < 2$, the behavior may be acceptable because the potential also diverges with the same power index, so the system is still bounded.

4. CENTRAL BLACK HOLE

We have found one further example of a cusp slope–anisotropy relationship in the literature. White (1981) found a relation⁴ by studying the special case of scale-free densities in scale-free potentials, namely

$$\gamma \geq \frac{\delta}{2} + \beta(2 - \delta), \quad (28)$$

where δ is the central power-law index for the potential (i.e., $\psi \sim r^{-\delta}$), which may be externally imposed. Note the change of the notation from White (1981). The form of the limit given in equation (28) is actually that of de Bruijne et al. (1996), who performed a similar study to White (1981) but allowed for flattening. This result was derived from a specific functional form for the DF, and in particular assumed that $f_E(E)$ is scale-free. We note that, since $\beta \leq 1$, the inequality $\gamma \geq 2\beta$ is automatically satisfied if $\gamma \geq 2$. For $\gamma < 2$, the self-consistent potential-density has a finite central potential, so $\delta = 0$. In this case, equation (28) reduces to $\gamma \geq 2\beta$. However, in the presence of a central black hole ($\delta = 1$), equation (28) provides us with a different limit $\gamma \geq \beta + \frac{1}{2}$, which is stricter than $\gamma \geq 2\beta$ if $\beta < \frac{1}{2}$.

Here we note that this limit can, without assuming that the density or DF is scale-free, be derived from the nonnegativity of the DF for a massless tracer population in the Keplerian potential of a central point mass. Let us suppose that the DF for

a massless tracer population is given by equation (1). (The results can then be extended to more general DFs using the identical argument of § 2.2.) Next, we assume that these tracers are subject to the potential $\psi = GM/r$ of a point mass at the center. Then the number density of the tracer population n can be found from integration of the DF over velocity space as in equation (2);

$$\begin{aligned} n &= \frac{D_\beta}{r^{2\beta}} \int_0^{GM/r} \left(\frac{GM}{r} - E \right)^{1/2-\beta} f_E(E) dE \\ &= \frac{D_\beta}{r^{\beta+1/2}} \int_0^{GM/r} (GM - rE)^{1/2-\beta} f_E(E) dE. \end{aligned} \quad (29)$$

For $\beta < \frac{1}{2}$, we find that

$$\frac{d}{dr} (r^{\beta+1/2} n) = -\tilde{D}_\beta \int_0^{GM/r} \frac{E f_E(E) dE}{(GM - rE)^{1/2+\beta}} \leq 0 \quad (30)$$

for any nonnegative DF. Consequently,

$$\frac{d}{dr} (r^{\beta+1/2} n) = r^{\beta-1/2} n \left(\beta + \frac{1}{2} + \frac{d \ln n}{d \ln r} \right) \leq 0 \quad (31)$$

or, equivalently,

$$-\frac{d \ln n}{d \ln r} \geq \beta + \frac{1}{2} \quad (32)$$

which is the desired result. In fact, if the external potential is replaced by $\psi = C/r^\delta$ in equation (29), it is easy to see that equation (28) simply follows from an essentially identical argument with $\gamma = -(d \ln n / d \ln r)$ being the central power index for the number density of the tracer population.

In fact, the result can still be obtained without assuming the specific form of (scale-free) potential. That is, provided only that $\delta = -\lim_{r \rightarrow 0} (d \ln \psi / d \ln r)$ and $0 \leq \delta \leq 1$, we find from equation (2) that

$$n = D_\beta \frac{\psi^{1/2-\beta}}{r^{2\beta}} \int_0^\psi \left(1 - \frac{E}{\psi} \right)^{1/2-\beta} f_E(E) dE, \quad (33)$$

and thus for $\beta < \frac{1}{2}$, we have

$$\frac{d}{d\psi} (nr^{2\beta} \psi^{\beta-1/2}) = \tilde{D}_\beta \int_0^\psi \frac{E f_E(E) dE}{\psi^2 [1 - (E/\psi)]^{1/2+\beta}} \geq 0. \quad (34)$$

However, of course,

$$\begin{aligned} \frac{d}{d\psi} (nr^{2\beta} \psi^{\beta-1/2}) &= \frac{nr^{2\beta-1} \psi^{\beta-1/2}}{d\psi/dr} \frac{d}{d \ln r} \ln (nr^{2\beta} \psi^{\beta-1/2}) \\ &= \frac{nr^{2\beta} \psi^{\beta-3/2}}{d \ln \psi / d \ln r} \left[\frac{d \ln n}{d \ln r} + 2\beta + \left(\beta - \frac{1}{2} \right) \frac{d \ln \psi}{d \ln r} \right] \geq 0, \end{aligned} \quad (35)$$

and by taking the limit $r \rightarrow 0$,

$$\gamma \geq 2\beta + \left(\frac{1}{2} - \beta \right) \delta. \quad (36)$$

Here note that $\lim_{r \rightarrow 0} (d \ln \psi / d \ln r) = -\delta \leq 0$. This also indicates that the result is still valid even if the self-gravity of the tracer population is appreciable, as long as the potential (that may be decoupled from the density) is divergent at the center.

If $\beta = \frac{1}{2}$, the limit given in equation (28) is, in fact, identical to $\gamma \geq 2\beta = 1$. Since the derivation of the limit in § 2 for $\beta = \frac{1}{2}$ does not use the assumption of self-consistency, it is still applicable here. Therefore, the limits in equations (32) and (36) can be extended to $\beta \leq \frac{1}{2}$. For $\frac{1}{2} < \beta \leq 1$, the limit $\gamma \geq 2\beta$ is actually stronger than equation (28) (note that $0 \leq \delta \leq 1$).

⁴ The inequality given in White (1981) or de Bruijne et al. (1996) did not include the case of equality because of the specific form of the scale-free DF. The general result actually extends to include the case of equality through the transition of the power-law distribution to the Dirac delta distribution.

However, with a centrally diverging potential (i.e., $\psi_0 = \infty$), the proof given in § 2.1.2 is not directly applicable. Instead, from equation (13), we now find that $\lim_{r \rightarrow 0} k$, where $k = nr^{2\beta}\psi^{\beta-1/2}$ either is finite and nonzero or diverges to positive infinity. Following exactly the same argument as in § 2.1.2 applied to $k = nr^{2\beta}\psi^{\beta-1/2}$ instead of $h = \rho r^{2\beta}$, we have

$$\lim_{r \rightarrow 0} \frac{d \ln k}{d \ln r} = \lim_{r \rightarrow 0} \frac{d \ln n}{d \ln r} + 2\beta + \left(\beta - \frac{1}{2}\right) \lim_{r \rightarrow 0} \frac{d \ln \psi}{d \ln r} \leq 0, \quad (37)$$

which translates to equation (36), and thus we can extend the limit of equation (36) to $\beta > \frac{1}{2}$ as well.

The limit $\gamma \geq \beta + \frac{1}{2}$ for $\delta = 1$ indicates that a spherical isotropic system subject to a Keplerian potential should possess a central density cusp at least as steep as $r^{-1/2}$. Similarly, if an isotropic stellar system is subject to a divergent dark matter potential due to a cusped profile with a slope steeper than that of the isothermal cusp, the stellar system should also be cusped with its cusp slope constrained to be $\gamma_\star \geq (\gamma_{\text{DM}}/2) - 1$, where γ_\star is the cusp index for the stellar system and $\gamma_{\text{DM}} \geq 2$ is that of the dark matter profile. Of course, if $\gamma_{\text{DM}} < 2$, the central potential is finite provided that there is no other source of divergent potential, and thus the limit simply reduces to $\gamma_\star \geq 0$. On the other hand, if the system were mildly radially anisotropic (near the center), that is to say $\beta \approx \frac{1}{2}$, the limit for the supportable cusp slope would be steeper, much like $\gamma \gtrsim 1$.

5. CONCLUSIONS

We have proved, for a very wide class of steady-state gravitating system, a theorem constraining the central cusp slope

of the density profile γ (eq. 20) and the central velocity anisotropy β . Specifically, the inequality $\gamma \geq 2\beta$ is a necessary condition for the nonnegativity of the distribution function (DF). If there is a divergent external potential, decoupled from the density profile, then the inequality generalizes to $\gamma \geq 2\beta - (\beta - \frac{1}{2})\delta$. Here the external potential diverges as $\sim r^{-\delta}$ at the center. Finally, if the external potential is due to a central black hole, it reduces to $\gamma \geq \beta + \frac{1}{2}$. We expect our result to be useful in the study of dense stellar systems, or in the building of extreme stellar dynamical models.

As most N-body simulations predict only very modest anisotropies ($\beta \approx 0$) in the very center, the application of our result does not directly constrain the central density profile ($\gamma \geq 0$). While the inequality derived by Hansen (2004), namely, $1 + \beta \leq \gamma < 3$, appears to be stronger than our result, his lower limit is only strictly applicable to the scale-free power-law density profile of infinite extent. It appears that $\gamma \geq \beta + 1$ actually constrains the asymptotic behavior of the density power index and the anisotropy parameter at infinity rather than at the center.

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APPENDIX

Here we provide a more detailed argument for the generality of the theorem than that given in § 2.2. While we do not claim that the following argument strictly adheres to the high standard of the pure mathematician, we hope that it indicates the generality of the result.

First, let us suppose that a DF $f(E, L)$ can be written as

$$f(E, L) = L^{-2\beta_0} [f_0(E) + f_1(E, L)], \quad (1)$$

where $f_0(E)$ is a function of E alone, whereas $f_1(E, L)$ is a continuous function that satisfies

$$f_1(E, L = 0) = 0, \quad (2)$$

which further implies that $f_0(E) \geq 0$ for all accessible values of E from the nonnegativity of the DF. Then

$$\begin{aligned} \rho &= 2\pi \iint L^{-2\beta_0} [f_0(E) + f_1(E, L)] v_T dv_T dv_r \\ &= \frac{D_{\beta_0}}{r^{2\beta_0}} \int_0^\psi dE (\psi - E)^{1/2-\beta_0} f_0(E) + \frac{4\pi}{r^{2\beta_0}} \iint dv_T dv_r v_T^{1-2\beta_0} f_1 \left(\psi - \frac{v_T^2 + v_r^2}{2}, rv_T \right) \end{aligned} \quad (3)$$

$$\begin{aligned} \rho \langle v_r^2 \rangle &= 2\pi \iint v_r^2 L^{-2\beta_0} [f_0(E) + f_1(E, L)] v_T dv_T dv_r \\ &= \frac{2D_{\beta_0}}{r^{2\beta_0}(3-2\beta_0)} \int_0^\psi dE (\psi - E)^{3/2-\beta_0} f_0(E) + \frac{4\pi}{r^{2\beta_0}} \iint dv_T dv_r v_T^{1-2\beta_0} v_r^2 f_1 \left(\psi - \frac{v_T^2 + v_r^2}{2}, rv_T \right) \end{aligned} \quad (4)$$

$$\begin{aligned} \rho \langle v_T^2 \rangle &= 2\pi \iint v_T^2 L^{-2\beta_0} [f_0(E) + f_1(E, L)] v_T dv_T dv_r \\ &= \frac{4D_{\beta_0}(1-\beta_0)}{r^{2\beta_0}(3-2\beta_0)} \int_0^\psi dE (\psi - E)^{3/2-\beta_0} f_0(E) + \frac{4\pi}{r^{2\beta_0}} \iint dv_T dv_r v_T^{3-2\beta_0} f_1 \left(\psi - \frac{v_T^2 + v_r^2}{2}, rv_T \right). \end{aligned} \quad (5)$$

Taking the limit $r \rightarrow 0$, the velocity moment integrals of the f_1 term vanishes, provided that the domain over which it is integrated is bounded and that f_1 is sufficiently well behaved. Note then that the anisotropy parameter at the center for this model is found to be

$$\lim_{r \rightarrow 0} \beta = 1 - \lim_{r \rightarrow 0} \frac{\rho \langle v_T^2 \rangle}{2\rho \langle v_r^2 \rangle} = 1 - (1 - \beta_0) = \beta_0. \quad (6)$$

Next, suppose that $h_0 = r^{2\beta_0} \rho_0$ and ρ_0 is the density profile built by the DF, $f(E, L) = L^{-2\beta_0} f_0(E)$. Then the proof given in § 2.1 indicates that $\gamma_0 \geq 2\beta_0$, where

$$\gamma_0 = -\lim_{r \rightarrow 0} \frac{d \ln \rho_0}{d \ln r}. \quad (7)$$

Now, since $\lim_{r \rightarrow 0} (d \ln h_0 / d \ln r) = 2\beta_0 - \gamma_0 \leq 0$, we have $\lim_{r \rightarrow 0} h_0 \neq 0$, that is, $\lim_{r \rightarrow 0} h_0$ is nonzero finite or diverges. Subsequently, from equation (3),

$$\lim_{r \rightarrow 0} (r^{2\beta_0} \rho - h_0) = \lim_{r \rightarrow 0} \left(\frac{r^{2\beta_0} \rho}{h_0} - 1 \right) h_0 = 0, \quad (8)$$

but $\lim_{r \rightarrow 0} h_0 \neq 0$, and therefore we find that

$$\lim_{r \rightarrow 0} \frac{r^{2\beta_0} \rho}{h_0} = 1. \quad (9)$$

If $\lim_{r \rightarrow 0} h_0$ is finite (i.e. $\gamma_0 = 2\beta_0$), then equation (9) implies that

$$0 < \lim_{r \rightarrow 0} r^{2\beta_0} \rho = \lim_{r \rightarrow 0} h_0 < \infty, \quad (10)$$

so $\lim_{r \rightarrow 0} r^{2\beta_0} \rho$ is also finite, that is, $\rho \sim r^{-2\beta_0}$. On the other hand, if $\lim_{r \rightarrow 0} h_0 = \infty$, l'Hôpital's rule indicates that

$$\lim_{r \rightarrow 0} \frac{r^{2\beta_0} \rho}{h_0} = \lim_{r \rightarrow 0} \frac{d(r^{2\beta_0} \rho)/dr}{dh_0/dr} = \lim_{r \rightarrow 0} \frac{r^{2\beta_0} \rho}{h_0} \frac{d \ln(r^{2\beta_0} \rho)/d \ln r}{d \ln h_0/d \ln r} = \frac{1}{2\beta_0 - \gamma_0} \left(2\beta_0 - \lim_{r \rightarrow 0} \frac{d \ln \rho}{d \ln r} \right) = 1. \quad (11)$$

In other words, combining the results in equations (10) and (11), we have established that

$$\lim_{r \rightarrow 0} \frac{d \ln \rho}{d \ln r} = \gamma_0 = \lim_{r \rightarrow 0} \frac{d \ln \rho_0}{d \ln r}, \quad (12)$$

where ρ is the density profile built by the DF of equation (1) and ρ_0 by $f(E, L) = L^{-2\beta_0} f_0(E)$. Finally, taking equations (6) and (12) together, we have established that the theorem $\gamma_0 > 2\beta_0$ extends to a more general class of DFs of the form of equation (1).

How general is the form of the DF in equation (1)? We argue that it is almost always possible to reduce most well-behaving DFs to the form of equation (1). That is, for a general DF of a spherically symmetric system in equilibrium, the reduction is possible if there exists $\alpha < 2$ such that

$$f_E(E) = \lim_{L \rightarrow 0} L^\alpha f(E, L), \quad (13)$$

where $f_E(E)$ should be finite and nonzero for values of E at least in some nonempty subset of all the accessible values of $E \in [0, \psi_0]$. Then the original DF can be written in a form of equation (1) as

$$f(E, L) = L^{-\alpha} [f_E(E) + f_1(E, L)]; \quad f_1(E, L) = L^\alpha f(E, L) - f_E(E). \quad (14)$$

and it is obvious to show that

$$f_1(E, L = 0) = \lim_{L \rightarrow 0} L^\alpha f(E, L) - f_E(E) = 0. \quad (15)$$

For example, for the DF of the form given by Cuddeford (1991),

$$f(E, L) = L^{-2\beta_0} f_0(Q); \quad Q \equiv E - \frac{L^2}{2r_a^2}, \quad (16)$$

the reduction is given by

$$f(E, L) = L^{-2\beta_0} [f_0(E) + f_1(E, L)]; \quad f_1(E, L) = f_0(Q) - f_0(E). \quad (17)$$

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